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LETTER TO THE EDITOR

On essential spectra of hard-core type Schrödinger operators

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Abstract. Using elementary methods such as Dirichlet decoupling, the computation of essential spectra of hard-core type Schrödinger operators $H^{\mathrm{D}}_{\Omega_2}$ in $L^2(\Omega_2)$ ($\Omega_2=\mathbb{R}^n\backslash\bar{\Omega}_1$, $\bar{\Omega}_1$ the compact hard core, $n\in\mathbb{N}$) is reduced to that of ordinary Schrödinger Hamiltonians in $L^2(\mathbb{R}^n)$ whose potentials coincide with that of $H^{\mathrm{D}}_{\Omega_2}$ sufficiently far out.

In this letter we assume

(i) $\Omega_1\subset\mathbb{R}^n$ is open and bounded, the boundary $\partial\Omega_1$ is of Lebesgue-measure zero. $\Omega_2:=\mathbb{R}^n\backslash\bar{\Omega}_1$. There is a $J\in C^\infty(\mathbb{R}^n)$ with

$$0\leqslant J\leqslant 1,\qquad J\restriction\bar{\Omega}_1=0,\qquad J(x)=1\quad\text{ for } |x|\geqslant R,$$
$$\bar{\Omega}_1\subseteq B_R=\{x\in\mathbb{R}^n\,|\,|x|<R\}. \tag{1}$$

If $-\Delta^{\mathrm{D}}_{\Omega_j}$ on $D(-\Delta^{\mathrm{D}}_{\Omega_j})=H^{2,2}_0(\Omega_j)$ denotes the Dirichlet Laplacian in $L^2(\Omega_j)$, $j=1,2$, we also use the assumption

(ii) $V\in L^1_{\mathrm{loc}}(\Omega_2)$ real-valued, $V\restriction\Omega_1=0$, $V_-(x):=\max\{0,-V(x)\}$ is form-bounded with respect to $-\Delta^{\mathrm{D}}_{\Omega_2}$ with bound $a<1$:

$$\|V_-^{1/2}g\|^2_{\Omega_2}\leqslant a\|(-\Delta^{\mathrm{D}}_{\Omega_2})^{1/2}g\|^2_{\Omega_2}+b\|g\|^2_{\Omega_2},\qquad g\in H^{2,1}_0(\Omega_2). \tag{2}$$

Then the form sum $H^{\mathrm{D}}_{\Omega_2}$ of $-\Delta^{\mathrm{D}}_{\Omega_2}$ and V in $L^2(\Omega_2)$, i.e.

$$H^{\mathrm{D}}_{\Omega_2}=-\Delta^{\mathrm{D}}_{\Omega_2}\dot{+}V \tag{3}$$

represents the so-called hard-core Hamiltonian. Under the natural isomorphism $L^2(\mathbb{R}^n)\simeq L^2(\Omega_1)\oplus L^2(\Omega_2)$ we have a decoupling of the Dirichlet Laplacian on $\Omega_1\cup\Omega_2$:

$$\Delta^{\mathrm{D}}_{\Omega_1\cup\Omega_2}=\Delta^{\mathrm{D}}_{\Omega_1}\oplus\Delta^{\mathrm{D}}_{\Omega_2}. \tag{4}$$

Its form domain $H^{2,1}_0(\Omega_1\cup\Omega_2)\simeq H^{2,1}_0(\Omega_1)\oplus H^{2,1}_0(\Omega_2)$ obviously is contained in the form domain $H^{2,1}(\mathbb{R}^n)$ of the Laplacian Δ in $\mathbb{R}^n$ and

$$Q^{-\Delta}\restriction H^{2,1}_0(\Omega_1\cup\Omega_2)=Q^{-\Delta^{\mathrm{D}}_{\Omega_1\cup\Omega_2}},\qquad Q^{-\Delta}\restriction H^{2,1}_0(\Omega_j)=Q^{-\Delta^{\mathrm{D}}_{\Omega_j}},\qquad j=1,2 \tag{5}$$

where $Q^{-\Delta},\ldots,$ denote the forms associated with $-\Delta,\ldots.$ For all $f\in H^{2,1}(\mathbb{R}^n)$ and all $\tilde{J}\in C^\infty(\mathbb{R}^n)$, $0\leqslant\tilde{J}\leqslant 1$, supp $\tilde{J}\subseteq\Omega_2$, we get

$$\tilde{J}f\in H^{2,1}_0(\Omega_2)\quad\text{and}\quad \|(-\Delta^{\mathrm{D}}_{\Omega_2})^{1/2}(\tilde{J}f)\|^2_{\Omega_2}\leqslant\|(-\Delta)^{1/2}f\|^2_{L^2}+\|\tilde{J}\Delta\tilde{J}\|_{L^\infty}\|f\|^2_{L^2}. \tag{6}$$

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This implies in particular that the form sum H_J of $-\Delta$ and J^2V in $L^2 = L^2(\mathbb{R}^n)$ is a well defined self-adjoint operator.

$$H_J = -\Delta \dotplus J^2 V. \tag{7}$$

Finally, for technical reasons, we introduce the form sum H^{D} in L^2:

$$H^{\mathrm{D}} = -\Delta^{\mathrm{D}}_{\Omega_1 \cup \Omega_2} \dotplus V \simeq (-\Delta^{\mathrm{D}}_{\Omega_1}) \oplus H^{\mathrm{D}}_{\Omega_2}. \tag{8}$$

As Ω_1 is bounded it follows for the essential spectra that

$$\sigma_{\mathrm{ess}}(H^{\mathrm{D}}) = \sigma_{\mathrm{ess}}(H^{\mathrm{D}}_{\Omega_2}). \tag{9}$$

The main result then reads

Theorem 1. Assumptions (i) and (ii) imply

$$\sigma_{\mathrm{ess}}(H^{\mathrm{D}}_{\Omega_2}) = \sigma_{\mathrm{ess}}(H_J). \tag{10}$$

Remarks

(a) Theorem 1 states in particular that the essential spectrum of H_J is independent of the choice of J satisfying (1). This is exactly what one expects on physical grounds: only the asymptotic behaviour of V (i.e. J^2V) determines the location of the essential spectrum; in particular the hard core can be thought of as a local singularity which is irrelevant for $\sigma_{\mathrm{ess}}(H^{\mathrm{D}}_{\Omega_2})$. This result represents a sort of 'decoupling from local singularities' as reviewed e.g. in Reed and Simon (1979) and Perry (1983) in the context of scattering theory in the presence of strongly singular potentials and in Hunziker (1967), Amrein and Georgescu (1973), Ferrero *et al* (1974), Rauch and Taylor (1975), Davies and Simon (1978), Kato (1978), Jensen and Kato (1978), Combes and Weder (1981), and Demuth (1982) in connection with scattering theory for hard core systems.

(b) Several (in fact infinitely many) hard cores are obviously included in our formulation by taking Ω_1 to be the union of disjoint open sets.

(c) We emphasise that the proof of theorem 1 as given below is completely elementary and only relies on well known compactness results.

Proof

(a) By Weyl's theorem (Reed and Simon 1978), (10) follows from (9) by showing that the compactness of

$$(H^{\mathrm{D}} - z)^{-1} - (H_J - z)^{-1}$$
$$= (H^{\mathrm{D}} - z)(1 - J) + [(H^{\mathrm{D}} - z)^{-1} J - J(H_J - z)^{-1}] + (1 - J)(H_J - z)^{-1} \tag{11}$$

for some $z \in \rho(H^{\mathrm{D}}) \cap \rho(H_J)$. But we have

$$(1 - J)(H^{\mathrm{D}} - \bar{z})^{-1} = [(1 - J)(|\nabla| + 1)^{-1}][(|\nabla| + 1)(H^{\mathrm{D}} - \bar{z})^{-1}], \tag{12}$$

$$(1 - J)(H_J - z)^{-1} = [(1 - J)(|\nabla| + 1)^{-1}][(|\nabla| + 1)(H_J - z)^{-1}]. \tag{13}$$

Since $1 - J \in C_0^\infty(\mathbb{R}^n)$, $(1 - J)(|\nabla| + 1)^{-1}$ is compact (Simon 1979). From Simon (1978) and Reed and Simon (1978) we know

$$-\Delta \leq (-\Delta^{\mathrm{D}}_{\Omega_1}) \oplus (-\Delta^{\mathrm{D}}_{\Omega_2}) = -\Delta^{\mathrm{D}}_{\Omega_1 \cup \Omega_2}. \tag{14}$$

Therefore $(|\nabla| + 1)(H^{\mathrm{D}} - \bar{z})^{-1}$ is bounded. Thus the first and the third summand of the RHS of (11) are compact operators.

(b) In order to prove the compactness of the second summand of the RHS of (11) we first note the following elementary fact on the difference of resolvents without proof.

Lemma 1. Let J be a bounded and A_j, $j=1,2$, be self-adjoint operators in a complex separable Hilbert space $\mathscr{H}$ and $z \in \rho(A_1) \cap \rho(A_2)$. Then

$$(A_2-z)^{-1}J - J(A_1-z)^{-1} = -(A_2-z)^{-1}B(A_1-z)^{-1} \tag{15}$$

for some linear operator $B: \mathscr{D}(A_1) \to \mathscr{H}$ iff

$$g \in \mathscr{D}(A_1) \qquad \text{implies } Jg \in \mathscr{D}(A_2) \text{ and}$$

$$A_2 Jg - JA_1 g = Bg \text{ for all } g \in \mathscr{D}(A_1). \tag{16}$$

And in order to determine the operator B in the present case we prove:

Lemma 2. Assume conditions (i) and (ii). Then $f \in \mathscr{D}(H_J)$ implies

(a) $Jf \in \mathscr{D}(H_J) \cap \mathscr{D}(H^D)$ and

(b) $H^D(Jf) = H_J(Jf) = J(H_J f) - 2(\nabla J)\cdot\nabla f - (\Delta J)f.$ (17)

Combining (15)-(17) we get

$$\begin{aligned}(H^D-z)^{-1}J - J(H_J-z)^{-1} &= (H^D-z)^{-1}[2(\nabla J)\cdot\nabla + (\Delta J)](H_J-z)^{-1}\\ &= 2[(|\nabla|+1)(H^D-\bar z)^{-1}]^*[(|\nabla|+1)^{-1}(\nabla J)]\cdot[\nabla(H_J-z)^{-1}]\\ &\quad + (H^D-z)^{-1}(\Delta J)(H_J-z)^{-1}\end{aligned} \tag{18}$$

which proves compactness of this operator in the same way as in (12) resp. (13). Thus the difference of the resolvents is compact and Weyl's theorem proves (10).

Proof of Lemma 2. Clearly, $Jf \in H_0^{2,2}(\Omega_1 \cup \Omega_2)$ and hence $Jf \in \mathscr{D}(H_J) \cap \mathscr{D}(|H^D|^{1/2})$. Denote by ∇ and by ∇^D the distributional gradient on $\mathscr{D}(\nabla) = H^{2,1}(\mathbb{R}^n)$ and on $\mathscr{D}(\nabla^D) = H^{2,1}(\Omega_1 \cup \Omega_2)$ respectively, and by Q^D and Q_J the forms associated with H^D and H_J. Let $g \in C_0^\infty(\mathbb{R}^n)$ then

$$\begin{aligned}Q_J(g, Jf) &= [\nabla g, \nabla(Jf)] + [|V|^{1/2}g, (\text{sign } V)|V|^{1/2}Jf]\\ &= [\nabla(Jg), \nabla f] - 2[(\nabla J)g, \nabla f] - [(\Delta J)g, f] + [|V|^{1/2}g, (\text{sign } V)|V|^{1/2}Jf]\\ &= (g, JH_J f) - 2[(\nabla J)g, \nabla f] - [(\Delta J)g, f]\end{aligned} \tag{19}$$

proves the right equality in equation (12) since $C_0^\infty(\mathbb{R}^n)$ is a form core for H_J. Next let $h \in C_0^\infty(\Omega_1 \cup \Omega_2)$ and compute (note $\nabla^D(Jf) = \nabla(Jf)$):

$$\begin{aligned}Q^D(h, Jf) &= (\nabla^D h, \nabla^D(Jf)) + (|V|^{1/2}h, (\text{sign } V)|V|^{1/2}(Jf))\\ &= (\nabla(Jh), \nabla f) - (h, (\Delta J)f) - 2(h, (\nabla J)\cdot\nabla f) + (|V|^{1/2}Jh, (\text{sign } V)|V|^{1/2}f)\\ &= (h, H_J(Jf)).\end{aligned} \tag{20}$$

Since $C_0^\infty(\Omega_1 \cup \Omega_2)$ is a form core for H^D (Cycon 1981) we get (17).

Thus the computation of $\sigma_{ess}(H^D_{\Omega_2})$ is reduced to that of $\sigma_{ess}(H_J)$. However criteria to determine the essential spectrum of Hamiltonians of the type H_J are well known (Reed and Simon 1978, Benci and Fortunato 1981, Leinfelder 1983).

We finally discuss generalisations of Weyl's theorem and their applications to singular (possibly hard-core type) potentials.

Lemma 3. Let A_j, $j = 1, 2$ be self-adjoint operators in some (complex, separable) Hilbert space $\mathscr{H}$ and $z \in \rho(A_1) \cap \rho(A_2)$. Suppose $J \in \mathscr{B}(\mathscr{H})$ and assume

(i) $(A_2 - z)^{-1}J - J(A_1 - z)^{-1} \in \mathscr{B}_\infty(\mathscr{H})$,

(ii) $(1 - J)(A_1 - z)^{-1} \in \mathscr{B}_\infty(\mathscr{H})$,

(iii) $(1 - J^*J)(A_1 - z)^{-1} \in \mathscr{B}_\infty(\mathscr{H})$.

Then (i) and (ii) as well as (i) and (iii) imply

$$\sigma_{\text{ess}}(A_1) \subseteq \sigma_{\text{ess}}(A_2).$$

(Here $\mathscr{B}(\mathscr{H})$ and $\mathscr{B}_\infty(\mathscr{H})$ denote the spaces of bounded and compact linear operators respectively in $\mathscr{H}$.)

Proof. It suffices to prove $\sigma_{\text{ess}}[(A_1 - z)^{-1}]\backslash\{0\} \subseteq \sigma_{\text{ess}}[(A_2 - z)^{-1}]\backslash\{0\}$.

Let $\mu \neq 0$, $\mu \in \sigma_{\text{ess}}[(A_1 - z)^{-1}]$ then $\mu = (\lambda - z)^{-1}$ for some $\lambda \in \sigma_{\text{ess}}(A_1)$. Thus we infer the existence of a singular sequence $\{g_m\}_{m \in \mathbb{N}} \subset \mathscr{H}$ for A_1 and λ such that $g_m \to^{w}_{m\to\infty} 0$, $\lim_{m\to\infty} \|g_m\| \geqslant \delta > 0$, $E_{A_1}[(a, b)]g_m = g_m$, $m \in \mathbb{N}$ ($(a, b) \subset \mathbb{R}$ bounded, E_{A_1} the spectral projection of A_1) and $(A_1 - \lambda)g_m \to^{s}_{m\to\infty} 0$ (Colgen 1981). Then $f_m = (A_1 - z)g_m$, $m \in \mathbb{N}$, is a singular sequence of $(A_1 - z)^{-1}$ and μ. This simply follows from:

$$f_m = (A_1 - \lambda)g_m + (\lambda - z)g_m \xrightarrow[m\to\infty]{w} 0,$$

$$\lim_{m\to\infty} \|f_m\| \geqslant |\lambda - z_0| \lim_{m\to\infty} \|g_m\| \geqslant |\lambda - z|\delta,$$

$$[(A_1 - z)^{-1} - \mu]f_m = -\mu(A_1 - \lambda)g_m \xrightarrow[m\to\infty]{s} 0.$$

In addition Jf_m, $m \in \mathbb{N}$, is a singular sequence of $(A_2 - z)^{-1}$ and μ: obviously $Jf_m \to^{w}_{m\to\infty} 0$ and $[(A_2 - z)^{-1} - \mu]Jf_m = [(A_2 - z)^{-1}J - J(A_1 - z)^{-1}]f_m + J[(A_1 - z)^{-1} - \mu]f_m \to^{s}_{m\to\infty} 0$.

Finally, from

$$Jf_m = f_m - [(1 - J)(A_1 - z)^{-1}](A_1 - z)^2 E_{A_1}[(a, b)]g_m,$$

we infer from (ii)

$$\lim_{m\to\infty} \|Jf_m\| = \lim_{m\to\infty} \|f_m\| \geqslant |\lambda - z|\delta.$$

Similarly, from

$$\|Jf_m\|^2 = \|f_m\|^2 + (f_m, [(J^*J - 1)(A_1 - z)^{-1}](A_1 - z)^2 E_{A_1}[(a, b))g_m],$$

we get from (iii)

$$\lim_{m\to\infty} \|Jf_m\| = \lim_{m\to\infty} \|f_m\| \geqslant |\lambda - z|\delta$$

proving $\mu \in \sigma_{\text{ess}}[(A_2 - z)^{-1}]$.

The above result under conditions (i) and (iii) has been obtained by Ginibre (1980) using methods different from singular sequences (cf also Colgen (1981) for related results). We emphasise that a symmetric version of lemma 3 using conditions (i) and (ii) adding $(1 - J^*)(A_2 - z)^{-1} \in \mathscr{B}_\infty(\mathscr{H})$ has been used in the proof of theorem 1 (identifying $\mathcal{H} = L^2(\mathbb{R}^n)$, J, $A_1 = H_J$, $A_2 = H^D$). But this symmetric variant is obviously equivalent to Weyl's theorem ($(A_2 - z)^{-1} - (A_1 - z)^{-1} \in \mathcal{B}_\infty(\mathcal{H}) \Rightarrow \sigma_{ess}(A_1) = \sigma_{ess}(A_2)$). Similarly, using conditions (i) and (iii) adding $(1 - JJ^*)(A_2 - z)^{-1} \in \mathcal{B}_\infty(\mathcal{H})$ in lemma 3 guarantees $\sigma_{ess}(A_1) = \sigma_{ess}(A_2)$. However in the context of theorem 1 these conditions again imply the hypothesis of Weyl's theorem after replacing J by J^2 in the definition of H_J. On the other hand the asymmetric versions of lemma 3 under certain circumstances are sufficient to yield invariance of the essential spectrum: if in addition to conditions (i) and (ii) or (i) and (iii) one has $\sigma_{ess}(A_1) = [c, \infty)$ and $A_2 \geqslant c$ for some $c \in \mathbb{R}$ then obviously $\sigma_{ess}(A_1) = \sigma_{ess}(A_2)$.

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